

RIEMANNIAN AND LORENTZIAN VOLUME COMPARISONS WITH THE BAKRY-EMERY RICCI TENSOR

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ABSTRACT. The Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor in a metric measure space are studied by the comparisons of the Jacobi differential equations in a Riemannian and Lorentzian manifold.

1. Introduction

Let M be an $(n+1)$ -dimensional complete and simply connected Riemannian manifold with metric g . Given a real valued smooth function f on M and Riemannian volume density $d\text{vol}_g$, a triple $(M, g, e^{-f} d\text{vol}_g)$ is called a metric measure space or a weighted manifold. The Bakry-Emery Ricci tensor Ric_f is defined by

$$(1.1) \quad \text{Ric}_f = \text{Ric} + \text{Hess}f,$$

which becomes the Ricci tensor if f is constant. In connection with the m -Bakry-Emery Ricci tensor defined by

$$\text{Ric}_f^m = \text{Ric} + \text{Hess}f - \frac{1}{m} df \otimes df, \quad \text{for } 0 < m \leq \infty,$$

the Bakry-Emery Ricci tensor $\text{Ric}_f = \text{Ric}_f^\infty$ is also called the ∞ -Bakry-Emery Ricci tensor. When $\text{Ric}_f = \lambda g$ for some constant λ , we have a gradient Ricci soliton which is an important topic in Ricci flow. The diffusion operator on a complete metric measure space via Bakry-Emery Ricci tensor has geometrical applications ([1] [12]). The Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor have been studied in [10], [11]. Let $\bar{M}(k)$ be an $(n+1)$ -dimensional Riemannian manifold of constant curvature k . Given a smooth function

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$h(x) = -a \cdot d(x, \bar{p})$ for $\bar{p} \in \bar{M}$, a positive real number a and the Riemannian distance function d with respect to the metric \bar{g} on $\bar{M}(k)$, a quadruple $(\bar{M}(k), \bar{g}, e^{-h} d\text{vol}_{\bar{g}}, \bar{p})$ is called the pointed metric measure space. Wei and Wylie proved mean curvature comparison under the Bakry-Emery Ricci inequality $\text{Ric}_f \geq nk$ together with $f' > -a$ [11]. Using mean curvature comparison, they showed the Bishop-Gromov volume comparison in terms of the weighted volume. And Ruan presented the Bishop volume comparison theorem by using the m -Bakry-Emery Ricci tensor and weighted Laplacian theorem in [10].

In this paper, we use the f -Jacobi equation (2.9) and (∞, f) -Raychaudhuri equation for a Jacobi tensor along a geodesic introduced for the study of Lorentzian singularity theorems in [2]. Our motivation is that the volume comparisons with the Bakry-Emery Ricci tensor can also be proved by the comparisons of the f -Jacobi differential equations using Lemma 5 obtained by Eschenburg and O'Sullivan in [5]. We show the Bishop and Bishop-Gromov volume comparisons between level hypersurfaces of geodesic balls with the Bakry-Emery Ricci tensor. A Lorentzian volume by the language of K -distance wedge defined in [3] is used to define the Lorentzian weighted volume. The Lorentzian Bishop and Bishop-Gromov volume comparisons with the Bakry-Emery Ricci tensor are similarly stated by using the same methods in Riemannian geometry. From the viewpoint of physics, the Lorentzian f -volume expansion rate between spacelike hypersurfaces under the m -Bakry-Emery Ricci tensor condition was considered in [8]. Our Lorentzian f -volume comparison could also be applied to the relative f -volume comparisons (cf. [7] [8] [9] [13]).

2. Preliminaries

Let H be a hypersurface in an $(n+1)$ -dimensional Riemannian manifold M and γ be a unit-speed geodesic orthogonal to H at $\gamma(r)$. For a unit normal vector field N along H with $N_{\gamma(r)} = \gamma'(r)$ and $q \in H$, a mapping $\phi : I \times H \rightarrow M$ given by

$$\phi(t, q) = \exp(t - r)N_q,$$

where $t \in I = [r, r_1)$, is called a normal geodesic variation of γ along the hypersurface H [5]. For each fixed $q \in H$, let γ_q be the geodesic given by $\gamma_q(t) = \phi(t, q)$ and define $\phi_t : H \rightarrow M$ by $\phi_t(q) = \phi(t, q)$ for $q \in H$. We denote by S_{-N} the shape operator of the hypersurface H . An H -Jacobi tensor along γ is defined by

DEFINITION 2.1. Let γ be a unit-speed geodesic orthogonal to a hypersurface H at $\gamma(r)$ with $N_{\gamma(r)} = \gamma'(r)$. A smooth $(1, 1)$ tensor field $A : (\gamma')^\perp \rightarrow (\gamma')^\perp$ associated with ϕ is called an H -Jacobi tensor along γ if it satisfies

$$A'' + R(A, \gamma')\gamma' = 0, \quad \ker A \cap \ker A' = \{0\}, \quad A(r) = \text{Id}, \quad A'(r) = S_{-N},$$

where Id is the identity endomorphism of $(\gamma')^\perp$. A point $\gamma(t_0)$ for $t_0 \in (r, r_1)$ is called a focal point to H if $\det A(t_0) = 0$.

The shape operator $S_{-\gamma'(t)}$ of each level hypersurface H_t of H associated with ϕ is given by as in [5]

$$A'A^{-1}(t) = S_{-\gamma'(t)} = S_t.$$

We denote by $\theta(t) = \text{tr} S_t$ the mean curvature of H_t along $\gamma(t)$.

Put $B = A'A^{-1}$ for an H -Jacobi tensor A along γ , then we have

$$(2.1) \quad B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R_{\gamma'} - B \circ B,$$

where we put $R(A, \gamma')\gamma' = R_{\gamma'}A$. The mean curvature is also expressed as

$$\theta = \text{tr}(B) = \frac{(\det(A))'}{\det(A)}$$

and the shear tensor σ of A along γ is defined by

$$\sigma = B - \frac{\theta}{n}\text{Id}.$$

Note that a variation tensor field A associated with ϕ is a Lagrange tensor (Proposition 1 in [5]). So the vorticity $\frac{1}{2}(B - B^*)$ is zero, where $*$ denotes the adjoint. Taking the trace of (2.1), we get the Raychaudhuri equation

$$(2.2) \quad \theta' + \frac{\theta^2}{n} + \text{Ric}(\gamma', \gamma') + \text{tr}\sigma^2 = 0,$$

where $\text{Ric}(\gamma', \gamma') = \sum_{i=1}^n g(R(e_i, \gamma')\gamma', e_i)$ for an orthonormal basis $\{e_i\}_{i=1}^n$ of γ'^\perp .

Put $x = (\det A)^{\frac{1}{n}}$, then we have

$$(2.3) \quad x' = \frac{1}{n}x\theta, \quad x'' = \frac{1}{n}(\theta' + \frac{\theta^2}{n})x.$$

So we obtain the Jacobi equation by (2.2) and (2.3)

$$x'' + \frac{1}{n}(\text{Ric}(\gamma', \gamma') + \text{tr}\sigma^2)x = 0.$$

DEFINITION 2.2. [2] Let A be an H -Jacobi tensor along a geodesic γ . For a smooth function $f : M \rightarrow \mathbb{R}$, define $B_f = A'A^{-1} - \frac{1}{n}(f \circ \gamma)' \text{Id}$. The f -expansion θ_f , f -shear tensor σ_f of A along γ is defined by

$$\theta_f = \text{tr}(B_f), \quad \sigma_f = B_f - \frac{\theta_f}{n} \text{Id},$$

respectively.

For a measure $e^{-f} d\text{vol}_g$, the f -expansion θ_f of an H -Jacobi tensor along a geodesic γ can be also expressed as

$$(2.4) \quad \theta_f = \frac{(e^{-f} \det A)'}{e^{-f} \det A} = \theta - f'.$$

So we have

$$\sigma_f = B_f - \frac{\theta_f}{n} \text{Id} = A'A^{-1} - \frac{f'}{n} \text{Id} - \frac{\theta - f'}{n} \text{Id} = B - \frac{\theta}{n} \text{Id} = \sigma.$$

Note that

$$\text{Hess}f(\gamma', \gamma') = g(D_{\gamma'} \nabla f, \gamma') = \gamma' g(\nabla f, \gamma') = f''.$$

Differentiating (2.4), we have

$$(2.5) \quad \theta'_f = \theta' - \text{Hess}f(\gamma', \gamma').$$

Using $\sigma_f = \sigma$, (1.1) and (2.5), the Raychaudhuri equation (2.2) is changed to

$$\theta'_f = -\left(\frac{1}{n}\theta^2 + \text{Ric}(\gamma', \gamma') + \text{tr}\sigma^2\right) - \text{Hess}f(\gamma', \gamma') = -\frac{1}{n}\theta^2 - \text{Ric}_f(\gamma', \gamma') - \text{tr}\sigma_f^2.$$

So we have

$$(2.6) \quad \theta'_f + \frac{1}{n}\theta^2 + \text{Ric}_f(\gamma', \gamma') + \text{tr}\sigma_f^2 = 0.$$

Hence, by inserting (2.4) to the equation (2.6), we get the f -Raychaudhuri equation as in [2]

$$(2.7) \quad \theta'_f + \frac{1}{n}(\theta_f^2 + 2\theta_f f' + (f')^2) + \text{Ric}_f(\gamma', \gamma') + \text{tr}\sigma_f^2 = 0.$$

Now put $x_f = (e^{-f} \det A)^{\frac{1}{n}}$. Then we see

$$(2.8) \quad x'_f = \frac{1}{n}x_f\theta_f, \quad x''_f = \frac{1}{n}(\theta'_f + \frac{1}{n}\theta_f^2)x_f.$$

So we obtain the f -Jacobi equation by (2.7) and (2.8)

$$(2.9) \quad x_f'' + \frac{1}{n}(\text{Ric}_f(\gamma', \gamma') + \frac{2\theta_f f' + (f')^2}{n} + \text{tr}\sigma_f^2)x_f = 0.$$

3. Riemannian volume comparisons with the Bakry-Emery Ricci tensor

Let M be an $(n + 1)$ -dimensional Riemannian manifold and γ be a unit speed geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let $S(r_0)$ be a sphere of radius r_0 in $T_p M$. Put $S_p(r_0) = \exp_p S(r_0)$. A Riemannian volume between $S_p(r_0)$ and $S_p(r)$ is defined by

$$(3.1) \quad V_{r_0}(r) = \int_{r_0}^r \int_{S(1)} |\det A| \, dv dt$$

for $0 < r_0 < r < \text{inj}_{S(1)}(p)$, where dv is the volume element of $S(1)$ and $\text{inj}_{S(1)}(p) = \inf\{\text{cut}_v(p) | v \in S(1)\}$. The weighted volume between the geodesic ball $S_p(r_0)$ and $S_p(r)$ is defined by

$$(3.2) \quad V_{r_0}^f(r) = \int_{r_0}^r \int_{S(1)} |e^{-f} \det A| \, dv dt.$$

Let $\bar{M}(k)$ be an $(n + 1)$ -dimensional Riemannian manifold of constant curvature k as the model space of volume comparison and $\bar{\gamma}$ be a unit speed geodesic with $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = \bar{v}$. For a Jacobi tensor \bar{A} along $\bar{\gamma}$ with $\bar{A}(0) = 0$ and $\bar{A}'(0) = \text{Id}$, the Jacobi equation along a geodesic $\bar{\gamma}$ is given by

$$\bar{x}'' + k\bar{x} = 0.$$

In order to compare volumes, put $S_{\bar{p}}(r_0) = \exp_{\bar{p}} S(r_0)$ and assume a linear isometry

$$(3.3) \quad \iota : T_{\gamma(r_0)} S_p(r_0) \rightarrow T_{\bar{\gamma}(r_0)} S_{\bar{p}}(r_0)$$

such that $\iota(\gamma'(r_0)) = \bar{\gamma}'(r_0)$ and $\iota(E_i(r_0)) = \bar{E}_i(r_0)$ for an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_{\gamma(r_0)} S_p(r_0)$ and its parallel basis $\{E_1, E_2, \dots, E_n\}$ along γ with $E_i(r_0) = e_i$ for each i , furthermore $S_{\bar{p}}(r_0) = \exp_{\bar{\gamma}(r_0)} \circ \iota \circ \exp_{\gamma(r_0)}^{-1} S_p(r_0)$. The following Lemma is essential for our volume comparisons.

LEMMA 3.1. [5] *Suppose $u : \mathbb{R} \rightarrow \mathbb{R}$ is smooth. Let x, \bar{x} be a smooth function such that x, \bar{x} is a solution of the differential inequality $x'' + ux \leq 0, \bar{x}'' + u\bar{x} = 0$, respectively, with $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \leq \bar{x}'(t_0)$. Suppose that x and \bar{x} are both positive in some interval $[t_0, t)$. Let $s,$*

\bar{s} be the first positive zero of x, \bar{x} , respectively. Then $s \leq \bar{s}, x \leq \bar{x}$ on $[t_0, \bar{s}]$ and $\frac{x'}{x} \leq \frac{\bar{x}'}{\bar{x}}$ on $[t_0, s]$.

Proof. Put $h = \frac{x}{\bar{x}}$ and $g = h'\bar{x}^2 = x'\bar{x} - x\bar{x}'$. If $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \leq \bar{x}'(t_0)$, then

$$g(t_0) = x'(t_0)\bar{x}(t_0) - x(t_0)\bar{x}'(t_0) = (x'(t_0) - \bar{x}'(t_0))\bar{x}(t_0) \leq 0$$

and $g' = x''\bar{x} - x\bar{x}'' = (x'' + ux)\bar{x} \leq 0$. So $g \leq 0$, hence $h' \leq 0$. Since $h(t_0) = 1$, we see $h \leq 1$. Therefore $x \leq \bar{x}$. It follows from

$$\frac{x'}{x} - \frac{\bar{x}'}{\bar{x}} = \frac{x'\bar{x} - x\bar{x}'}{x\bar{x}},$$

$g(t_0) \leq 0, g' \leq 0$ that $\frac{x'}{x} \leq \frac{\bar{x}'}{\bar{x}}$. □

We denote by $\bar{V}_{r_0}^f(r), \bar{V}_{r_0}(r)$ the f -volume, volume between level hypersurfaces in Riemannian manifold $\bar{M}(k)$ of constant curvature k respectively (3.2), (3.1). Under the above notations, we have

PROPOSITION 3.2. *Let $\bar{M}(k)$ be an $(n + 1)$ -dimensional Riemannian manifold of constant curvature $k \geq 0$. Assume that $e^{-f(r_0)}\det A(r_0) = \det \bar{A}(r_0)$ and $\theta_f(r_0) \leq \bar{\theta}(r_0)$ for a complete metric measure space $(M, g, e^{-f} d\text{vol}_g)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk, \theta_f(r_0) \leq 0$ and $f' \leq 0$, then we get*

$$V_{r_0}^f(r) \leq \bar{V}_{r_0}(r), \quad \frac{V_{r_0}^f(R)}{V_{r_0}(R)} \leq \frac{V_{r_0}^f(r)}{\bar{V}_{r_0}(r)}$$

for $0 < r_0 < r < R < \text{inj}_{S(1)}(p)$. Equality holds if and only if each level hypersurface $S_p(t) = \exp_p S(t)$ is isometric to $S_{\bar{p}}(t) = \exp_{\bar{p}} S(t)$ for $r_0 \leq t < r$ and $f = 0$.

Proof. Assume that $e^{-f(r_0)}\det A(r_0) = \det \bar{A}(r_0)$ and $\theta_f(r_0) \leq \bar{\theta}(r_0)$ with the linear isometry (3.3) $\iota : T_{\gamma(r_0)}S_p(r_0) \rightarrow T_{\bar{\gamma}(r_0)}S_{\bar{p}}(r_0)$ such that $S_{\bar{p}}(r_0) = \exp_{\bar{\gamma}(r_0)} \circ \iota \circ \exp_{\gamma(r_0)}^{-1} S_p(r_0)$. From (2.6) and $\text{Ric}_f(\gamma', \gamma') \geq nk$, it follows that

$$\theta'_f = -\left(\frac{1}{n}\theta^2 + \text{Ric}_f(\gamma', \gamma') + \text{tr}\sigma_f^2\right) \leq -\text{Ric}_f(\gamma', \gamma') \leq -nk.$$

Integration gives

$$\int_{r_0}^t \theta'_f ds \leq - \int_{r_0}^t nk ds$$

$$\theta_f(t) \leq -nk(t - r_0) + \theta_f(r_0).$$

Thus we get $\theta_f(t) \leq 0$ under the assumption $\theta_f(r_0) \leq 0$. Furthermore we have $\theta_f(t)f'(t) \geq 0$, since we assume $f'(t) \leq 0$. From the f -Jacobi equation (2.9) along a geodesic γ , it follows that

$$\frac{x_f''(t)}{x_f(t)} \leq -\frac{1}{n}\text{Ric}_f(\gamma'(t), \gamma'(t)) - \frac{2}{n^2}\theta_f(t)f'(t) \leq -\frac{1}{n}\text{Ric}_f(\gamma'(t), \gamma'(t)),$$

since $\text{tr}\sigma_f^2 \geq 0$ and $\theta_f(t)f'(t) \geq 0$. Thus we get

$$(3.4) \quad \bar{x}'' + k\bar{x} = 0, \quad \frac{x_f''}{x_f} \leq -\frac{1}{n}\text{Ric}_f(\gamma', \gamma') \leq -k.$$

The inequality (3.4) under $e^{-f(r_0)}\det A(r_0) = \det \bar{A}(r_0)$ and $\theta_f(r_0) \leq \bar{\theta}(r_0)$ implies $x_f \leq \bar{x}$ by Lemma 3.1. So we obtain the volume inequality $V_{r_0}^f(r) \leq \bar{V}_{r_0}(r)$.

If the equality $V_{r_0}^f(r) = \bar{V}_{r_0}(r)$ holds, then we have $e^{-f}\det A = \det \bar{A}$. So $\bar{\theta} = \theta_f$. Recall that the Raychaudhuri equation in $\bar{M}(k)$ is

$$(3.5) \quad \bar{\theta}' + \frac{1}{n}\bar{\theta}^2 + nk = 0.$$

Since $\bar{\theta} = \theta_f$, we get by subtracting (3.5) from (2.7)

$$nk - \text{Ric}_f(\gamma', \gamma') = \frac{2\theta_f f'}{n} + \frac{(f')^2}{n} + \text{tr}\sigma_f^2.$$

By the assumption of $\text{Ric}_f(\gamma', \gamma') \geq nk$, we get $\sigma_f = \sigma = 0$ and $f' = 0$. Therefore we obtain $\bar{\theta} = \theta_f = \theta$. Then we have $\bar{B} = B$ and we get $R_{\gamma'} = k\text{Id}$ from (2.1). The conclusion follows from Theorem 4 in [6].

By Lemma 3.1, we get $\theta_f(t) \leq \bar{\theta}(t)$ for $r_0 \leq t < R$. The Bishop-Gromov volume comparison theorem follows from [4] (cf. [3]) under the assumptions $\theta_f(r_0) \leq \bar{\theta}(r_0)$ and $e^{-f(r_0)}\det A(r_0) = \det \bar{A}(r_0)$. If the equality of the Bishop-Gromov comparison holds, then we get $x_f = \bar{x}$ [4] (cf. [3]), that is $e^{-f}\det A = \det \bar{A}$. The above arguments lead to the conclusion. \square

Recall that given a smooth function $h(x) = -a \cdot d(x, \bar{p})$ for a positive real number a and $\bar{p} \in \bar{M}(k)$, where $\bar{M}(k)$ is an $(n + 1)$ -dimensional Riemannian manifold of constant curvature k and d is the Riemannian distance function in \bar{M} , a quadruple $(\bar{M}(k), \bar{g}, e^{-h}d\text{vol}_{\bar{g}}, \bar{p})$ is called

the pointed metric measure space. For the measure $e^{-h}d\text{vol}_{\bar{g}}$, the h -expansion is given by

$$\theta_h(t) = \frac{(e^{-h(t)} \det \bar{A}(t))'}{e^{-h(t)} \det \bar{A}(t)} = \bar{\theta}(t) + a,$$

where $\bar{\theta}(t)$ is the mean curvature of each level hypersurface $S_{\bar{p}}(t)$ along a geodesic $\bar{\gamma}$ orthogonal to a hypersurface $S_{\bar{p}}(r_0)$.

THEOREM 3.3. *Let $(\bar{M}(k), \bar{g}, e^{-h}d\text{vol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for $k \geq 0$. For a complete metric measure space $(M, g, e^{-f}d\text{vol}_g)$, assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)}\det A(r_0) = e^{-h(r_0)}\det \bar{A}(r_0)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \leq a$ and $f' \leq -a$, then we get*

$$V_{r_0}^f(r) \leq V_{r_0}^h(r), \quad \frac{V_{r_0}^f(R)}{V_{r_0}^h(R)} \leq \frac{V_{r_0}^f(r)}{V_{r_0}^h(r)}$$

for $0 < r_0 < r < R < \text{inj}_{S(1)}(p)$. Equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $r_0 \leq t < r$ and $f = h$.

Proof. Put $h_{\bar{p}}(x) = -a \cdot d(x, \bar{p})$, $h_p(x) = -a \cdot d(x, p)$ along a unit speed geodesic $\bar{\gamma}, \gamma$ in \bar{M}, M , respectively. Then we have

$$h_{\bar{p}}(t) = -a \cdot d(\bar{p}, t) = -at = -a \cdot d(p, t) = h_p(t).$$

For a smooth function $\tilde{f} = f - h_p$ on M , we get $\text{Ric}_{\tilde{f}}(\gamma', \gamma') = \text{Ric}_f(\gamma', \gamma') \geq nk$, since $h_p'' = 0$. We apply Proposition 3.2 for a complete metric measure space $(M, g, e^{-\tilde{f}}d\text{vol}_g)$ under the assumptions $\theta_{f-h_p}(r_0) \leq 0$ and $(f - h_p)' \leq 0$ which are equivalent to $\theta_f(r_0) \leq a$ and $f' \leq h_p' = -a$. With these initial conditions, we have by Lemma 3.1

$$(3.6) \quad x_{f-h_p} \leq \bar{x}, \quad \theta_{f-h_p} \leq \bar{\theta}.$$

The inequality (3.6) leads to

$$x_{f-h_p} = (e^{-(f-h_p)} \det A)^{\frac{1}{n}} \leq \bar{x}.$$

Hence we get

$$(3.7) \quad x_f = (e^{-f} \det A)^{\frac{1}{n}} \leq (e^{-h_p})^{\frac{1}{n}} (\det \bar{A})^{\frac{1}{n}} = (e^{-h_p} \det \bar{A})^{\frac{1}{n}} = x_h.$$

In the same way, we see that

$$\theta_{f-h_p} = \theta_{f-h_{\bar{p}}} = \theta - (f' + a) \leq \bar{\theta}$$

implies

$$(3.8) \quad \theta_f = \theta - f' \leq \bar{\theta} + a = \bar{\theta} - h_p' = \theta_h.$$

Therefore the volume inequality follows from (3.7) and (3.8). The volume equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $r_0 \leq t < r$ and $f = h$ by the same arguments of Proposition 3.2 with $\tilde{f} = f - h_p$. □

Recall that we denote by $V_{r_0}^f(r)$, $V_{r_0}(r)$ the f -volume, volume between level hypersurfaces, respectively (3.2), (3.1). We Put $V^f(r) = V_0^f(r)$ and $V(r) = V_{r_0}(r)$.

PROPOSITION 3.4. *Let $\bar{M}(k)$ be an $(n + 1)$ -dimensional Riemannian manifold of constant curvature $k \geq 0$. For a complete metric measure space $(M, g, e^{-f} d\text{vol}_g)$, assume that $\theta_f(r_0) \leq \bar{\theta}(r_0)$ and $e^{-f(r_0)} \det A(r_0) = \det \bar{A}(r_0)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \geq 0$ and $f' \geq 0$, then we get*

$$V^f(r_0) \leq \bar{V}(r_0), \quad \frac{V^f(R)}{\bar{V}(R)} \leq \frac{V^f(r)}{\bar{V}(r)}$$

where $r < R \leq r_0$. Equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $0 < t < r_0$ with $f = 0$.

Proof. Consider a Jacobi tensor A along a geodesic $\beta(t) = \gamma(r_0 - t)$ such that $A'A^{-1} = S_{-\beta'(t)}$. Then $\theta_f(t)$ along $\beta(t)$ for $0 \leq t \leq r_0$ satisfies (3.4). Integration gives

$$\int_t^{r_0} \theta'_f ds \leq - \int_t^{r_0} nk ds$$

$$\theta_f(r_0) - \theta_f(t) \leq -nk(r_0 - t).$$

Hence we get

$$nk(r_0 - t) + \theta_f(r_0) \leq \theta_f(t).$$

Therefore if $\theta_f(r_0) \geq 0$, then $\theta_f(t) \geq 0$. Under the assumption of $f' \geq 0$, we get $\theta_f(t)f'(t) \geq 0$. Hence the conclusions follow from Proposition 3.2 as t approaches zero. □

Apply Theorem 3.3 for a smooth function $\tilde{f} = f - h_p$ on M and $\theta_{\tilde{f}}$ along the geodesic $\beta(t) = \gamma(r_0 - t)$ for $0 \leq t \leq r_0$. Then we get by Proposition 3.4

THEOREM 3.5. *Let $\bar{M}(k)$ be an $(n + 1)$ -dimensional Riemannian manifold of constant curvature $k \geq 0$. For a complete metric measure space $(M, g, e^{-f} d\text{vol}_g)$, assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)} \det A(r_0) =$*

$e^{-h(r_0)}\det\bar{A}(r_0)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \geq a$ and $f' \geq -a$, then we get

$$V^f(r_0) \leq V^h(r_0), \quad \frac{V^f(R)}{V^h(R)} \leq \frac{V^f(r)}{V^h(r)}$$

for $0 < r < R < r_0 < \text{inj}_{S(1)}(p)$. Equality holds if and only if each level hypersurface $S_p(t)$ is isometric to $S_{\bar{p}}(t)$ for $0 < t < r_0$ with $f = h$.

4. Lorentzian volume comparisons with the Bakry-Emery Ricci tensor

Riemannian volume comparisons with the Bakry-Emery Ricci tensor can be applied very similarly in a Lorentzian manifold by using the *K-distance wedge* in [3]. We introduce it here for reader's convenience. Let M be an $(n + 1)$ -dimensional globally hyperbolic space-time and γ be a unit speed timelike geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Take an orthonormal basis $\{e_1, \dots, e_n, \gamma'(0)\}$ of T_pM and let E_i be the parallel field along γ such that $E_i(0) = e_i$ for each i . Consider a geodesic variation

$$\alpha(t, s) = \exp_p(t(v + sE_i))$$

along γ , then we have a Jacobi field $J_i(t) = \alpha_*|_{(t,0)}(\frac{d}{ds}) = (d\exp_p)_{tv}te_i$ such that $J_i(0) = 0$ and $J'_i(0) = e_i$ for each i . Let A be a Jacobi tensor along γ with the initial conditions $A(0) = 0$ and $A(0)' = \text{Id}$, then we obtain

$$t^n |\det \alpha_*|_{(t,0)}| = \|J_1(t) \wedge J_2(t) \wedge \dots \wedge J_n(t)\| = |\det A|.$$

Let $\text{Fut}(T_pM)$ be the set of all future directed timelike vectors $v \in T_pM$ such that $\exp_p(v)$ is defined and put $H(r_0) = \{v \in \text{Fut}(T_pM) | g(v, v) = -r_0^2\}$. Let \bar{K} be a compact subset of $H(1)$. Define the *K-distance wedge* $B_p^K(r)$ as

$$B_p^K(r) = \{\exp_p(tv) \mid v \in \bar{K}, 0 \leq t \leq r\}$$

and put $V^K(r) = \text{Vol}(B_p^K(r))$. Let du, dv be the volume element of $\text{Fut}(T_pM), \bar{K}$, respectively. The Lorentzian volume element is given by $du = t^n dv dt$ (Lemma 4.2 [3]) and

$$V(r) = V^K(r) = \int_0^r \int_{\bar{K}} |\det A| \, dv dt.$$

for $0 < r < \text{inj}_{\bar{K}}(p)$, where $\text{inj}_{\bar{K}}(p) = \inf\{\text{cut}_v(p) | v \in \bar{K}\}$. Using the comparison of the Jacobi differential equation (Lemma 3.1), the Lorentzian

version of the Bishop and Bishop-Gromov comparison theorems under $\text{Ric}(\gamma', \gamma') \geq nk$ are obtained in [3].

Now we show the Lorentzian version of the Bishop and Bishop-Gromov comparisons between level hypersurfaces with the Bakry-Emery Ricci tensor. Let M be an $(n + 1)$ -dimensional globally hyperbolic space-time and γ be a unit speed timelike geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Take a compact subset $H^*(r_0)$ of $H(r_0)$ and put $H_{r_0}^* = \exp_p H^*(r_0)$. The Lorentzian f -volume between level hypersurfaces $H_{r_0}^*$ and H_r^* is defined by

$$V_{r_0}^f(r) = \int_{r_0}^r \int_{\bar{K}} |e^{-f} \det A| \, dv dt$$

for $0 < r_0 < r < \text{inj}_{\bar{K}}(p)$.

Let $\bar{M}(-k)$ be an $(n + 1)$ -dimensional space-time of constant curvature $-k(k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic with $\bar{\gamma}(0) = \bar{p}$ and $\bar{\gamma}'(0) = \bar{v}$. For a Jacobi tensor \bar{A} along $\bar{\gamma}$ with $\bar{A}(0) = 0$ and $\bar{A}'(0) = \text{Id}$, the Jacobi equation along a geodesic $\bar{\gamma}$ is given by

$$\bar{x}'' + k\bar{x} = 0$$

as in [6]. For the Lorentzian distance function d , consider the pointed metric measure space $(\bar{M}(k), \bar{g}, e^{-h} d\text{vol}_{\bar{g}}, \bar{p})$, where $h : \exp_{\bar{p}}(\text{Fut}(T_{\bar{p}}\bar{M})) \rightarrow [0, \infty)$ and $h(x) = -a \cdot d(x, \bar{p})$. Put $\bar{H}_{r_0}^* = \exp_{\bar{p}} H^*(r_0)$ for a compact subset $H^*(r_0)$ of $H(r_0) \subset T_{\bar{p}}\bar{M}$. We assume a linear isometry $\iota : T_{\gamma(r_0)} H_{r_0}^* \rightarrow T_{\bar{\gamma}(r_0)} \bar{H}_{r_0}^*$ such that $\iota(\gamma'(r_0)) = \bar{\gamma}'(r_0)$ and $\iota(E_i(r_0)) = \bar{E}_i(r_0)$ for an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of $T_{\gamma(r_0)} H_{r_0}^*$ and its parallel basis $\{E_1, E_2, \dots, E_n\}$ along γ with $E_i(r_0) = e_i$ for each i , furthermore $\bar{H}_{r_0}^* = \exp_{\bar{\gamma}(r_0)}^{-1} \circ \iota \circ \exp_{\gamma(r_0)} H_{r_0}^*$.

All Riemannian volume comparisons with Bakry-Emery Ricci tensor obtained in the previous section can be stated similarly in a Lorentzian manifold. A Lorentzian manifold M is assumed to be globally hyperbolic so that the Lorentzian distance function is finite valued and continuous (cf. [3]).

THEOREM 4.1. *Let $(M, g, e^{-f} d\text{vol}_g)$ be a metric measure space for a globally hyperbolic space-time M and γ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $H_{r_0}^*$. Let $(\bar{M}(-k), \bar{g}, e^{-h} d\text{vol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for a space-time of constant curvature $-k(k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $\bar{H}_{r_0}^*$. Assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)} \det A(r_0) = e^{-h(r_0)} \det \bar{A}(r_0)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \leq a$ and*

$f' \leq h' = -a$, then we get $\theta_f \leq \theta + a = \theta_h$ and

$$V_{r_0}^f(r) \leq V_{r_0}^h(r), \quad \frac{V_{r_0}^f(R)}{V_{r_0}^h(R)} \leq \frac{V_{r_0}^f(r)}{V_{r_0}^h(r)},$$

where $R(> r)$ is less than the minimum of the focal values of $\bar{H}_{r_0}^*$ and $H_{r_0}^*$. Equality holds if and only if each level hypersurface H_t^* is isometric to \bar{H}_t^* for $r_0 \leq t < R$ and $f = h$.

THEOREM 4.2. *Let $(M, g, e^{-f} d\text{vol}_g)$ be a metric measure space for a globally hyperbolic space-time M and γ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $H_{r_0}^*$. Let $(\bar{M}(-k), \bar{g}, e^{-h} d\text{vol}_{\bar{g}}, \bar{p})$ be the pointed metric measure space for a space-time of constant curvature $-k(k > 0)$ and $\bar{\gamma}$ be a unit speed timelike geodesic orthogonal to a spacelike hypersurface $\bar{H}_{r_0}^*$. Assume that $\theta_f(r_0) \leq \theta_h(r_0)$ and $e^{-f(r_0)} \det A(r_0) = e^{-h(r_0)} \det \bar{A}(r_0)$. If $\text{Ric}_f(\gamma', \gamma') \geq nk$, $\theta_f(r_0) \geq a$ and $f' \geq -a$, then we get $\theta_f \leq \theta + a = \theta_h$ and*

$$V^f(r_0) \leq V^h(r_0), \quad \frac{V^f(R)}{V^h(R)} \leq \frac{V^f(r)}{V^h(r)},$$

where $r < R \leq r_0$. Equality holds if and only if each level hypersurface H_t^* is isometric to \bar{H}_t^* for $0 < t < r_0$.

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